A Transmission Line Model For The Lossless Ferrite-Loaded Nonreciprocal Waveguide

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In this paper, the lossless TE mode waveguide equivalent-circuit model is extended to permit a description of nonreciprocal phase shift effects by incorporation of distributed gyrators into the elemental line length prototype. The gyrators provide antireciprocal coupling between the series and shunt inductive elements of the transmission line model. Simple perturbational formulas are presented and used for computing some elementary geometries.

I. Introduction

Equivalent-circuit transmission line representations provide a simple means of describing the essential behavior of the propagation factor and characteristic impedance of a lossless guiding structure without demanding a detailed knowledge of the electromagnetic field distribution [1]. Some years ago, the distributed gyrator was incorporated into a coupled-transmission line ensemble and used to describe nonreciprocal Faraday rotation effects [2]. A variation of this model was subsequently used to verify that the reciprocal phase shift mechanism of the Reggia-Spencer type ferrite phase shifter was one of suppressed Faraday rotation, and to study the characteristics of such structures [3, 4].

The equivalent-circuit model developed here extends previous work by representing a single-mode, lossless, nonreciprocal transmission line. The feature that permits this behavior is the introduction of a distributed gyrator into the elemental line length prototype such that series and shunt inductive members are coupled nonreciprocally. If the "series inductance" and "shunt inductance" quantities are respectively associated with the transverse and longitudinal magnetic fields, it is clear that this model has coupling properties analogous to the nonreciprocal waveguide.

II. Elemental Prototype Of Line Length

Consider a uniformly filled ordinary rectangular waveguide and its dominant TE mode equivalent transmission line representation as defined in Figure 1. The characteristic impedance will be

$$Z_o = \sqrt{Z/Y} = \frac{j\omega\mu}{\gamma}\frac{b}{a} \qquad (1)$$

with the propagation factor γ given by

$$\gamma = \sqrt{ZY} = j\omega_c\sqrt{\mu\varepsilon}\sqrt{\left(\frac{\omega}{\omega_c}\right)^2 - 1} \qquad (2)$$

where $\omega_c = \frac{\pi}{a}\frac{1}{\sqrt{\mu\varepsilon}}$ is the cutoff frequency of the mode.

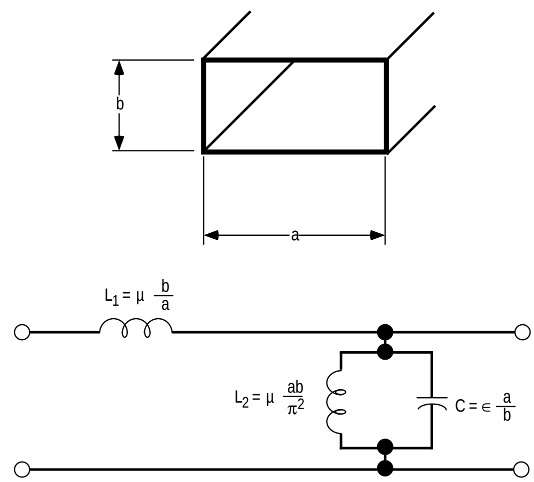


Fig. 1 Rectangular waveguide

Suppose now that the series and shunt inductance elements are coupled by means of a distributed gyrator. This can be done by writing an "incremental inductance matrix" $d\underline{L}$ as follows:

$$d\underline{L} = \frac{L_1}{k_c(1-\zeta^2)}\begin{bmatrix} k_c\ dz & -j\zeta \\ j\zeta & \dfrac{1}{k_c\ dz}\end{bmatrix} \qquad (3)$$

This quantity $d\underline{L}$ is an equivalent-circuit representation for the gyromagnetically coupled inductance values in an increment dz of the line length. The gyromagnetic coupling factor ζ can vary in magnitude between zero and κ/μ for a ferrite medium. The basis for writing such a matrix relationship has been discussed previously (2).

Referring to the gyrator-coupled equivalent circuit of Figure 2, it is possible to write the following equations relating line voltage and current:

$$\begin{bmatrix} dV \\ \\ V \end{bmatrix} = -\frac{j\omega L_1}{k_c\left(1-\zeta^2\right)} \begin{bmatrix} k_c dz & -j\zeta \\ j\zeta & \dfrac{1}{k_c dz} \end{bmatrix} \begin{bmatrix} I \\ \\ dI_2 \end{bmatrix} \qquad (4)$$

and

$$dI = j\omega C\, dz\, V + dI_2 \qquad (5)$$

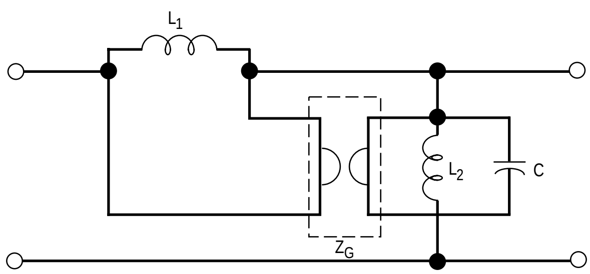


Fig. 2 Gyrator-coupled equivalent circuit

The second line of equation (4) can be solved for dI_2 in terms of V and I, and the result substituted into the top line of equation (4) and into equation (5). After some manipulation, a version of the telegrapher's equations is obtained:

$$\begin{bmatrix} \dfrac{dV}{dz} \\ \\ \dfrac{dI}{dz} \end{bmatrix} = -j \begin{bmatrix} k_c\zeta & \omega L_1 \\ \omega C\left[1-\left(\dfrac{\omega_c}{\omega}\right)^2\left(1-\zeta^2\right)\right] & k_c\zeta \end{bmatrix} \begin{bmatrix} V \\ \\ I \end{bmatrix} \qquad (6)$$

III. Propagation Factor And Characteristic Impedance

Assuming solutions for V and I that vary as $e^{\gamma z}$, equations (6) take the standard form of a characteristic-value problem, with solution for propagation factor as follows:

$$\gamma = -j\left[k_c\ \zeta \pm \omega\ \sqrt{L_1 C}\ \ \sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2\left(1-\zeta^2\right)}\right] \qquad (7)$$

Defining the initial (ζ=0), infinite-medium propagation factor as $\beta_o = \omega\sqrt{L_1 C}$, equation (7) can be written in the form

$$\gamma = j\beta_o\left[\zeta\left(\frac{\omega_c}{\omega}\right) \pm \sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2\left(1-\zeta^2\right)}\right] \qquad (8)$$

For the ζ=0, this expression obviously reduces to the form of equation (2) and describes the behavior of an ordinary waveguide. When ζ is nonzero, however, in the frequency region where $\omega>\omega_C$, the roots for γ split into positive and negative values that are unequal, corresponding to a guide that propagates a traveling wave in each direction, but at distinctly different phase velocities. At $\omega = \omega_C$, only one of the roots for γ vanishes, but this point no longer represents the boundary between propagating and nonpropagating frequency regions. Cutoff begins for

$$\omega_c' = \omega_c\ \sqrt{1-\zeta^2} \qquad (9)$$

and in the region $\omega' < \omega < \omega_c$, the roots for γ both have the same sign; i.e., two traveling waves can propagate in the same direction at different phase velocities. The phase velocities of these two waves become equal and finite at cutoff. Below cutoff, the propagation factor becomes complex, indicating that the solutions have the characteristic of attenuated waves.

Consider the region $\omega > \omega_c'$ and define $j\beta = \gamma$. Then the group velocity of the traveling waves can be found by forming the derivative of ω with respect to ß, and from equation (7) will be

$$V_g = \frac{d\omega}{d\beta} = \frac{1}{d\beta / d\omega} = \pm\frac{\sqrt{\omega^2 - \omega_c'^2}}{k_o} \qquad (10)$$

It is evident that the two traveling wave solutions have equal and opposite group velocity regardless of the magnitude or sign of their respective phase velocities. The specific dependence of the ω–β characteristic is plotted as a family of curves in Figure 3. The characteristic impedance values are the V/I ratios associated with each of the roots of γ, and are essentially the eigenvector ratios of the matrix on the right-hand side of equation (6).

$$Z_{o1} = -Z_{o2} = \sqrt{\frac{L_1}{C\left[1-\left(\dfrac{\omega_c'}{\omega_c}\right)^2\right]}} \qquad (11)$$

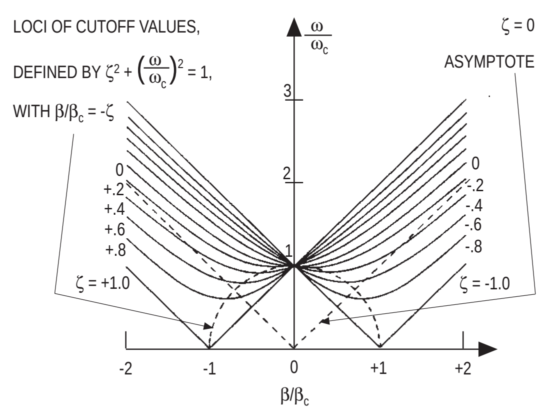


Fig. 3 Specific dependence of ω–β characteristics

IV. Coupling Factor Computation

The transmission line model described above characterizes the ferrite gyromagnetic effects by means of a single coupling factor, ζ. In this section, a perturbational formula is derived for computation of ζ for a very simple field distribution, using an approach similar to that presented by Hord, *et al* [4].

First, assume that the field distribution in the waveguide is principally TE, and represent the field components E_y, H_X and H_Z as follows:

$$bE_y = V(z)\, e_y(x);\ aH_X = I(z)\, h_X(x);\ bH_Z = I_S(z)\, h_Z(x) \tag{12}$$

Here $e_y(x)$, $h_X(x)$, and $h_Z(x)$ are transverse-plane distribution functions, and $V(z)$ and $I(z)$ are complex functions that vary in the direction of propagation and are associated in a general way with the "transmission line" voltage and current, while $I_S(z)$ is a complex function that can be associated with the "shunt inductive current". Using the permeability tensor for a transversely magnetized ferrite, it follows that

$$B_x = \mu H_x - j\kappa H_z;\quad B_z = j\kappa H_x + \mu H_z \tag{13}$$

where κ and μ are, in general, functions of position.

Now, write the time-harmonic Maxwell curl equations, substitute for B using equation (13), and write E and H in terms of $V(z)e_y(x)$, etc., using equation (12). The resulting system of equations is obtained:

$$\frac{1}{b}\frac{dV(z)}{dz}e_y(x) = j\omega\left[-\mu I(z)h_x(x) + jkI_s(z)h_z(x)\right] \tag{14}$$

$$\frac{1}{a}\frac{dI(z)}{dz}e_y(x) = j\omega_\varepsilon V(z)e_y(x) + I_s(z)\frac{dh_z(x)}{dx} \tag{15}$$

$$\frac{1}{b}V(z)\frac{de_y(x)}{dx} = j\omega\left[j\kappa I(z)h_x(x) + \mu I_s(z)h_z(x)\right] \tag{16}$$

Equation (16) and its derivative with respect to x may be used to eliminate $I_S(z)dh_Z(x)/dx$ in Equations (14) and (15), respectively. In taking the derivative, it should be noted that κ and μ may be functions of the transverse coordinate. The resulting equations are:

$$\frac{1}{b}\frac{dV}{dz}e_y(x) = j\frac{\kappa}{\mu}V\frac{de_y(x)}{dx} - j\omega\left(\frac{\mu^2-\kappa^2}{\mu}\right)Ih_x(x) \tag{17}$$

$$\frac{1}{a}\frac{dI}{dz}h_x = j\omega_\varepsilon\left[e_y(x) + \frac{1}{\omega 2_\varepsilon}\frac{d}{dx}\left(\frac{1}{\mu}\frac{de_y(x)}{dx}\right)\right]V - jI\frac{d}{dx}\left(\frac{\kappa}{\mu}h_x(x)\right) \tag{18}$$

Now multiply Equation (26) by $h_x^*(x)$ and the complex conjugate of the time-dependent part of (27) by $e_y(x)$, then integrate across the transverse dimension and write in matrix form:

$$\begin{bmatrix}\dfrac{dV}{dz}\\[2em]\dfrac{dI}{dz}\end{bmatrix} = -j\begin{bmatrix} -\dfrac{\int_o^a \frac{\kappa}{\mu}e_y' h_x^* dx}{\int_o^a e_y h_x^* dx} & \omega\dfrac{\int_o^a \frac{\mu^2-\mu^2|h_x|^2 dx}{\mu}}{\int_o^a e_y h_x^* dx}\\[2em] \omega\dfrac{\int_o^a \varepsilon\left\{e_y^* + \frac{1}{\omega 2_\varepsilon}\left[\frac{e_y'^*}{\mu}\right]\right\}e_y dx}{\int_o^a e_y h_x^* dx} & \dfrac{\int_o^a\left[\frac{\kappa}{\mu}h_x^*\right]' e_y dx}{\int_o^a e_y h_x^* dx}\end{bmatrix}\begin{bmatrix}V\\[2em] I\end{bmatrix} \tag{19}$$

Here the prime denotes differentiation with respect to x. Since $e_y(x)$ must vanish at x=o and x=a, integration by parts shows that the diagonal elements of Equation (19) are identically equal in conformity with Equation (6).

Assuming $e^{\gamma z}$ solutions for V(z) and I(z), Equation (19) becomes a characteristic-value problem analogous to Equation (6), with roots for γ given by:

$$\gamma = -j\frac{\left[\int_o^a \frac{\kappa}{\mu}e_y' h_x^* dx \pm \omega\sqrt{\left(\int_o^a\left(\frac{\mu^2-\kappa^2}{\mu}\right)|h_x|^2 dx\right)\left(\int_o^a \varepsilon\left\{e_y^* + \frac{1}{\omega^2\varepsilon}\left[\frac{e_y^{*\prime}}{\mu}\right]\right\}e_y dx\right)}\right]}{\int_o^a e_y h_x^* dx} \tag{20}$$

As a pertubational formula, the differential propagation factor will be given by

$$\Delta\beta = 2\frac{\int_o^a \frac{\kappa}{\mu}e_y' h_x^* dx}{\int_o^a e_y h_x^* dx} \tag{21}$$

For small deviations from the uniformly filled case, transverse resonance may be applied to conclude the following relationship:

$$\frac{1}{\omega^2\varepsilon} - \frac{d}{dx}\left[\frac{1}{\mu}\frac{de_y(x)}{dx}\right] = -\left(\frac{\omega_c'}{\omega}\right)^2 e_y(x) \tag{22}$$

Comparison with Equation (12) then permits the following identifications to be made:

$$L_1 = \frac{b}{a}\frac{\int_o^a\left(\frac{\mu^2-\kappa^2}{\mu}\right)|h_x|^2 dx}{\int_o^a e_y h_x^* dx}\quad;\quad C = \frac{a}{b}\frac{\int_o^a \varepsilon|e_y|^2 dx}{\int_o^a e_y h_x^* dx} \tag{23}$$

Since k_C is well defined for this case, the coupling factor is given by

$$\zeta = -\frac{1}{k_c}\frac{\int_o^a \frac{\kappa}{\mu}(x)e_y'(x)h_x^*(x)dx}{\int_o^a e_y(x)h_x^*(x)dx} \tag{24}$$

V. Applications Of The Method To Simple Examples

a. Completely filled waveguide, Figure 4(a) assume

$$e_y(x) = h_z(x) = \sin\frac{\pi x}{a}; \quad k_c = \frac{\pi}{a} \text{ so that}$$

$$\int_o^a e_y h_x^* dx = \int_o^a \sin^2\left(\frac{\pi x}{a}\right)dx = \frac{a}{2} \text{ and therefore}$$

$$\zeta = -\frac{2}{a}\int_o^a \frac{\kappa}{\mu}(x)\,\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}dx \qquad (25)$$

If $\kappa/\mu\,(x)$ is an even function about a/2, $\zeta = 0$. For maximum coupling, take κ/μ as negative for the left half of the guide, and positive for the right half. Then

$$\zeta = \frac{2}{a}\frac{\kappa}{\mu}\int_o^{\frac{a}{2}} \sin\frac{2\pi x}{a}dx = \frac{2}{\pi}\frac{\kappa}{\mu} \qquad (26)$$

The differential phase shift available from the completely filled guide can be expressed as the difference between the two roots of Equation (8), and will be

$$\frac{\Delta\phi}{\phi_o} = \frac{\Delta\beta}{\beta_o} = 2\zeta\frac{\omega_c}{\omega} = \frac{4}{\pi}\frac{\kappa}{\mu}\frac{\omega_c}{\omega} \qquad (27)$$

b. Thin vertical slab located at $x=x_o$, of thickness t<<a, Figure 4b. Again, assume

$$e_y(x) = h_x(x) = \sin\frac{\pi x}{a}\;;\; k_c = \frac{\pi}{a}$$

For a thin slab, approximate $\kappa/\mu\,(x)$ as

$$\frac{\kappa}{\mu}(x) \approx \frac{\kappa}{\mu}\; t\;\delta\;(x - x_o)$$

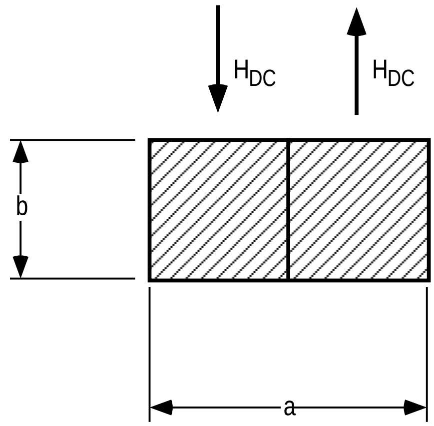
so that

$$\zeta \approx -\frac{t}{a}\frac{\kappa}{\mu}\,\sin\left(\frac{2\pi x_o}{a}\right) \qquad (28)$$

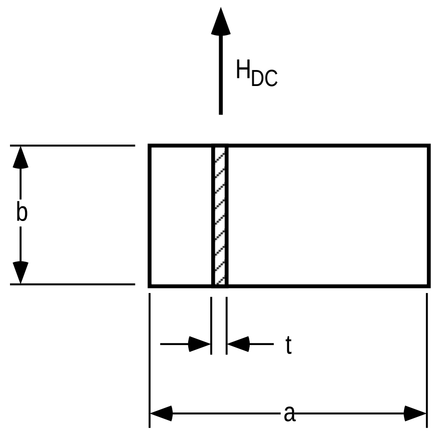
It appears that the optimum locations for such a thin slab are at $x_o = \frac{a}{4}$ and $x_o = \frac{3a}{4}$, regardless of frequency relative to cut-off. At these locations, differential phase shift will be

$$\frac{\Delta\phi}{\phi_o} = \frac{\Delta\beta}{\beta_o} \approx \frac{2t}{a}\frac{\kappa}{\mu}\frac{\omega_c}{\omega} \qquad (29)$$

for a single slab, and twice that amount for two slabs oppositely magnetized at the two positions. Again, a dispersive decrease of differential angle with frequency is indicated. It should be noted that this result is in agreement with computations based on conventional perturbation theory.



a. COMPLETELY FILLED WAVEGUIDE



b. WAVEGUIDE WITH THIN VERTICAL SLAB

Fig. 4 Waveguide cross-sectional geometries

VI. References

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